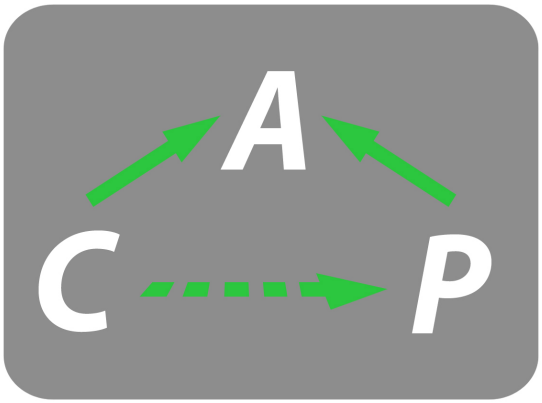
A constructive approach to finitely presented functors

Sebastian Posur

University of Siegen

September 28, 2017



Finitely presented functors

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Q: Is it possible to model the category of finitely presented functors on the computer?

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What are finitely presented functors?

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Let **A** be an abelian category

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Definition

A functor $F : \mathbf{A} \to \mathbf{Ab}$ is called **finitely presented** if there exist $A, B \in \mathbf{A}$ and an exact sequence of functors

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Applications in representation theory of finite dimensional algebras, algebraic geometry, model theory, homological algebra.

Route

Route

Finitely presented modules

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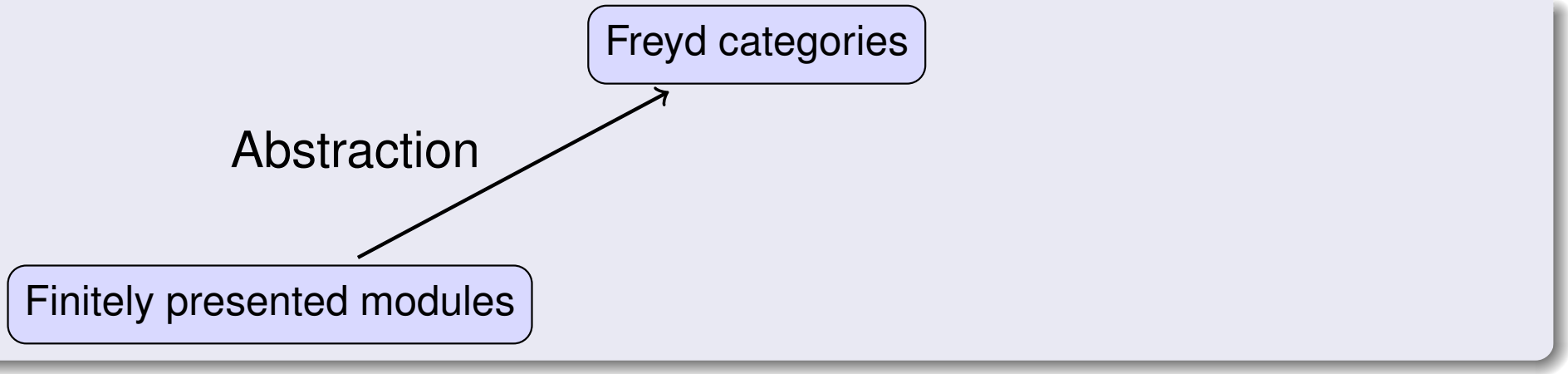
Freyd categories
Abstraction
Finitely presented modules

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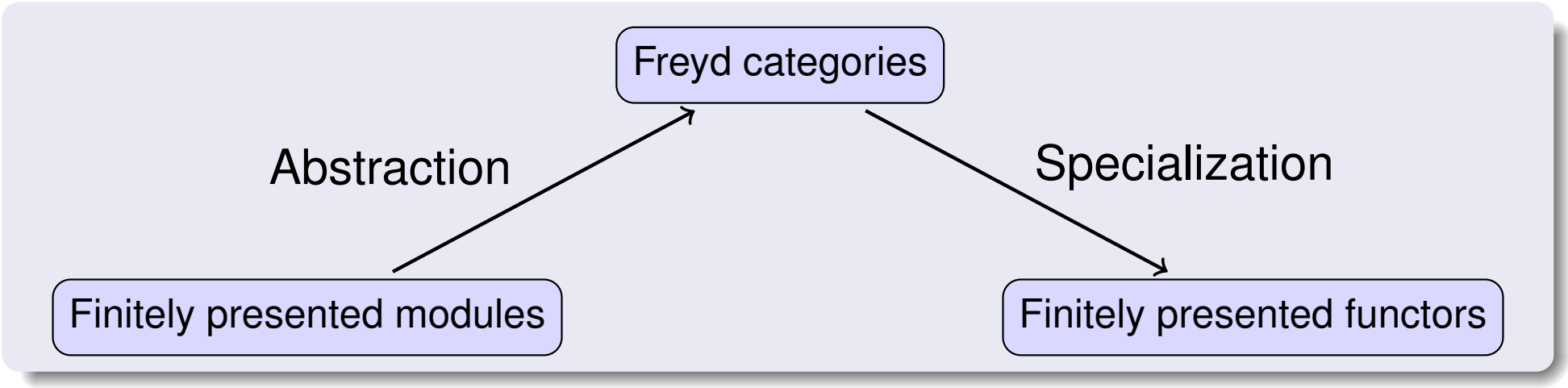
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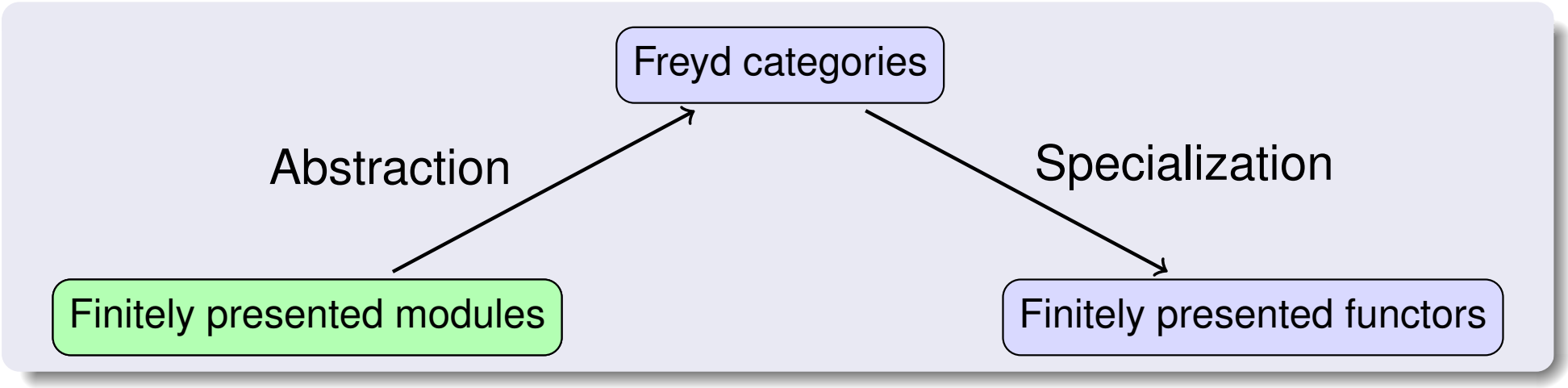
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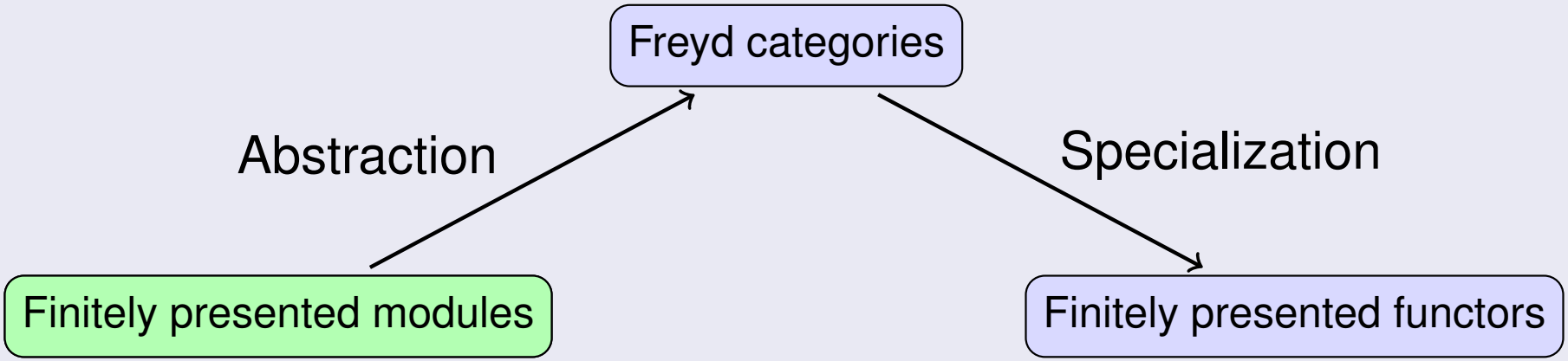

1. Data structures

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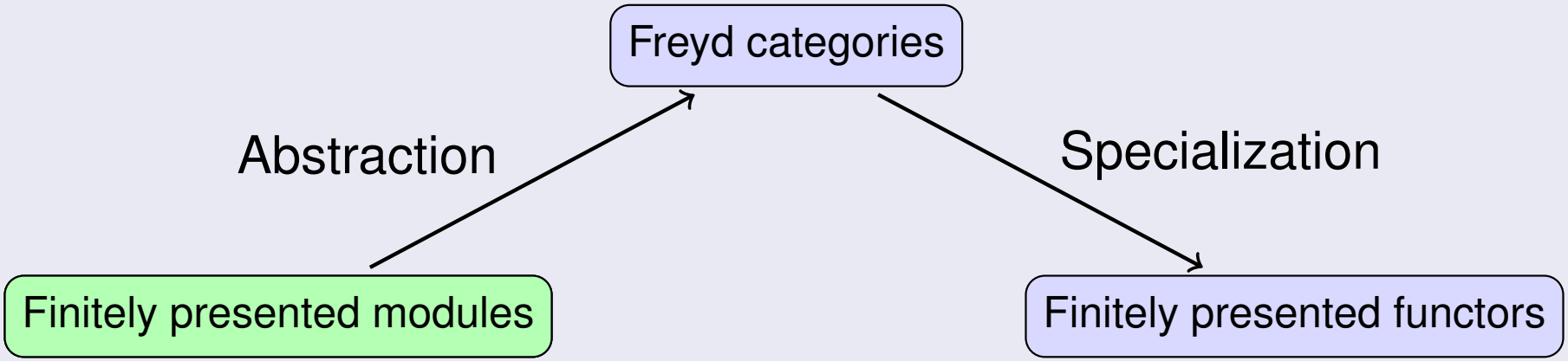

1. Data structures
2. Algorithms

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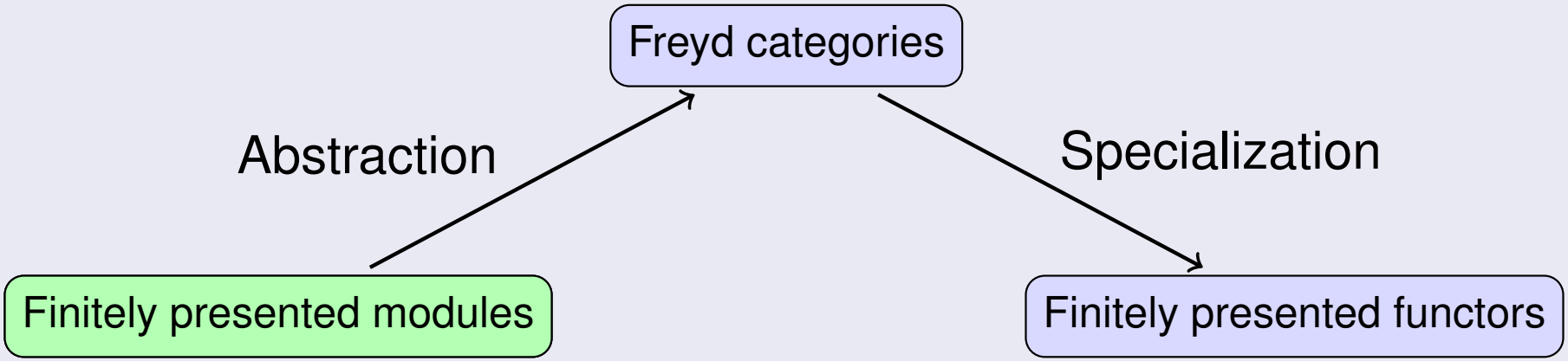

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Finitely presented modules: Data structures

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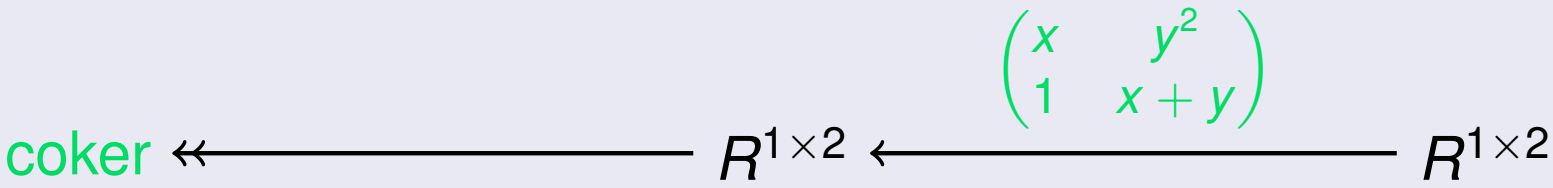

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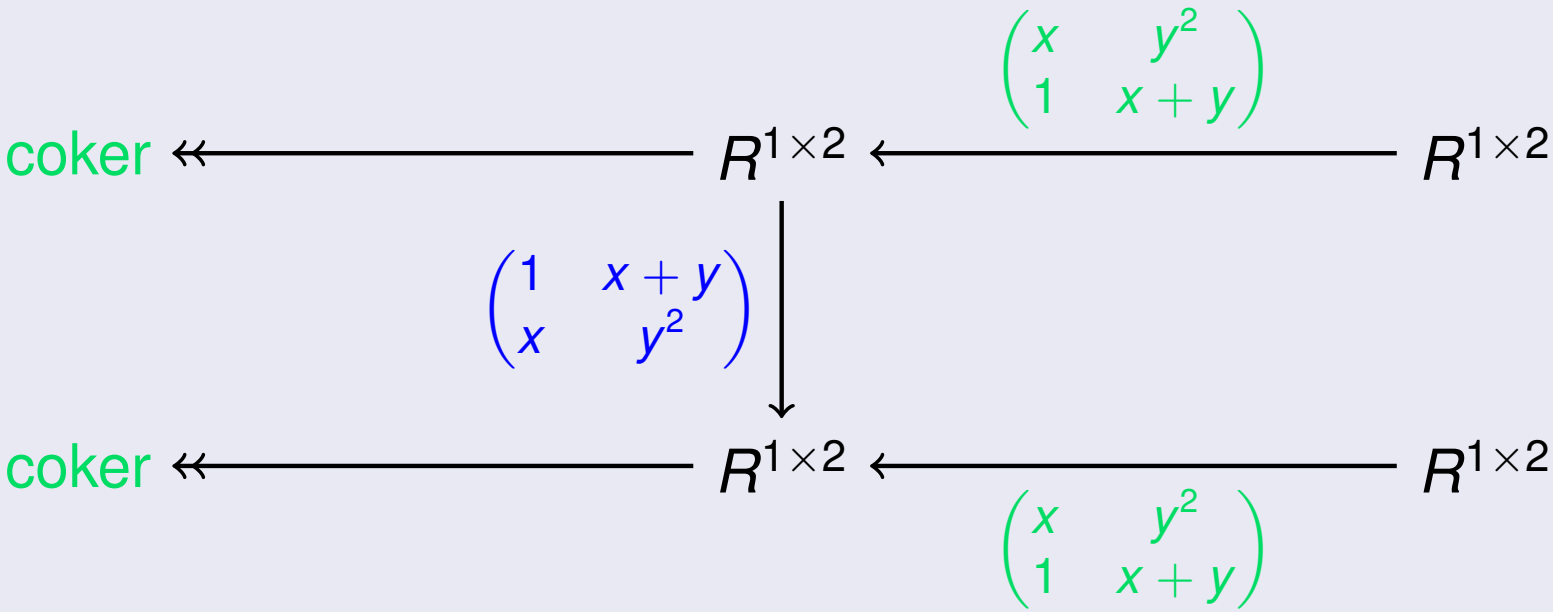

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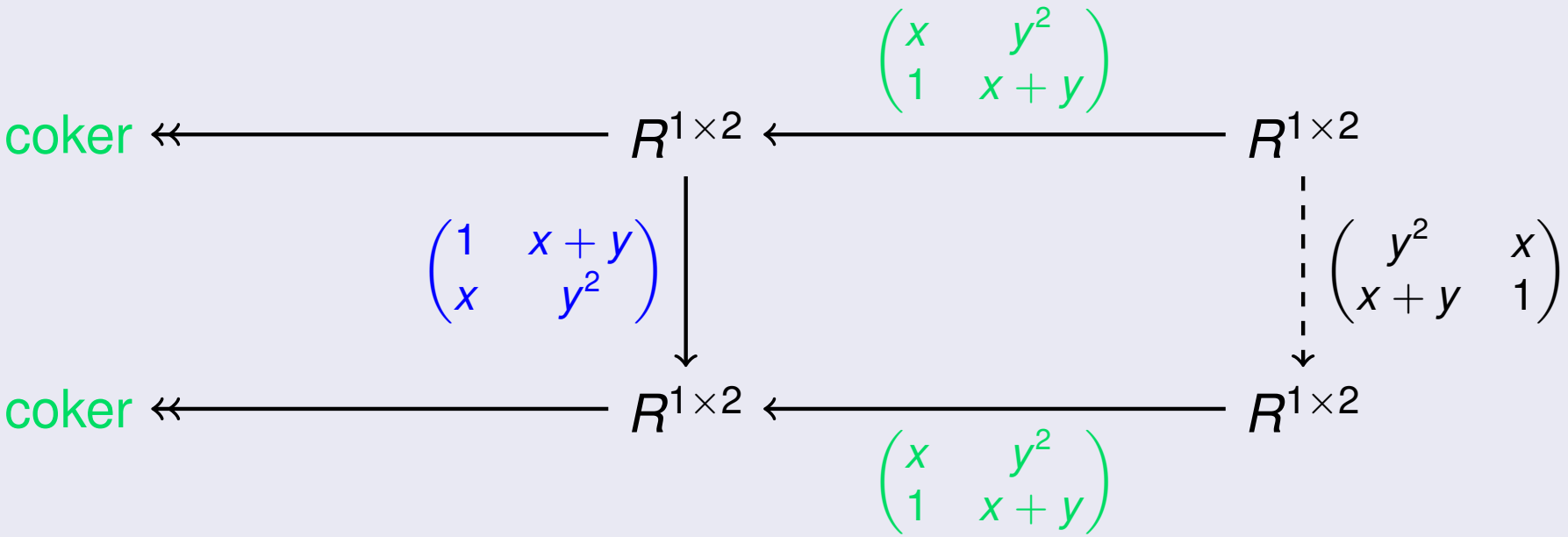

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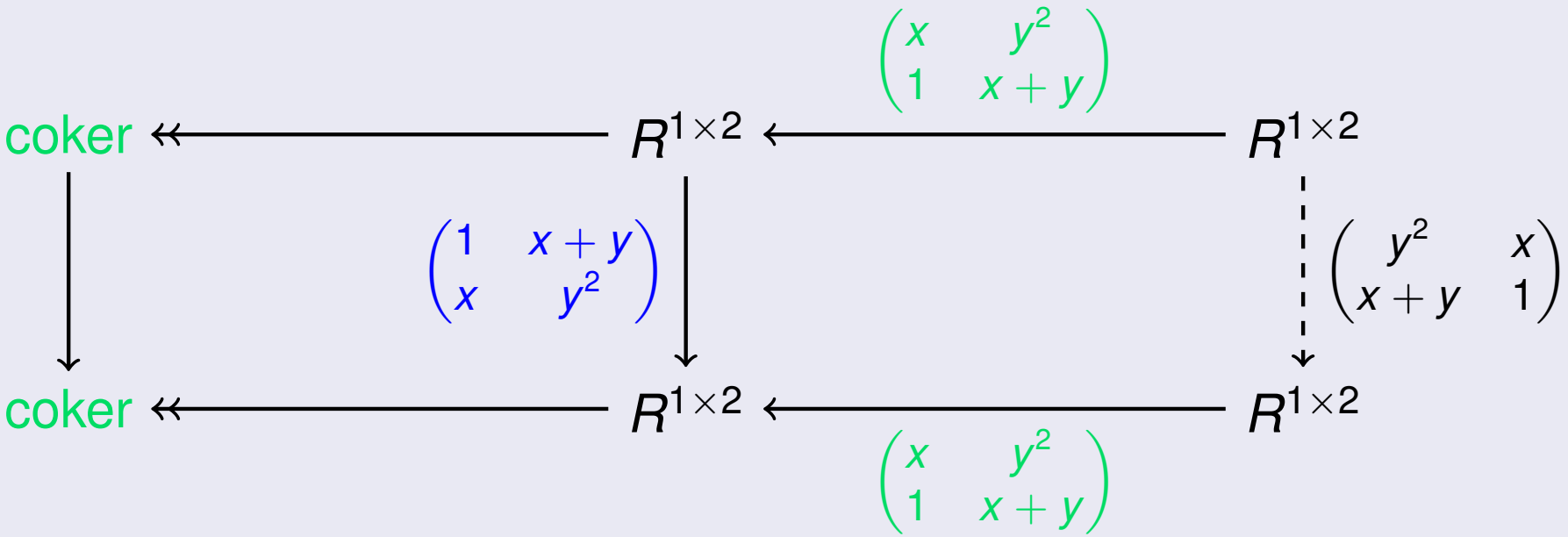

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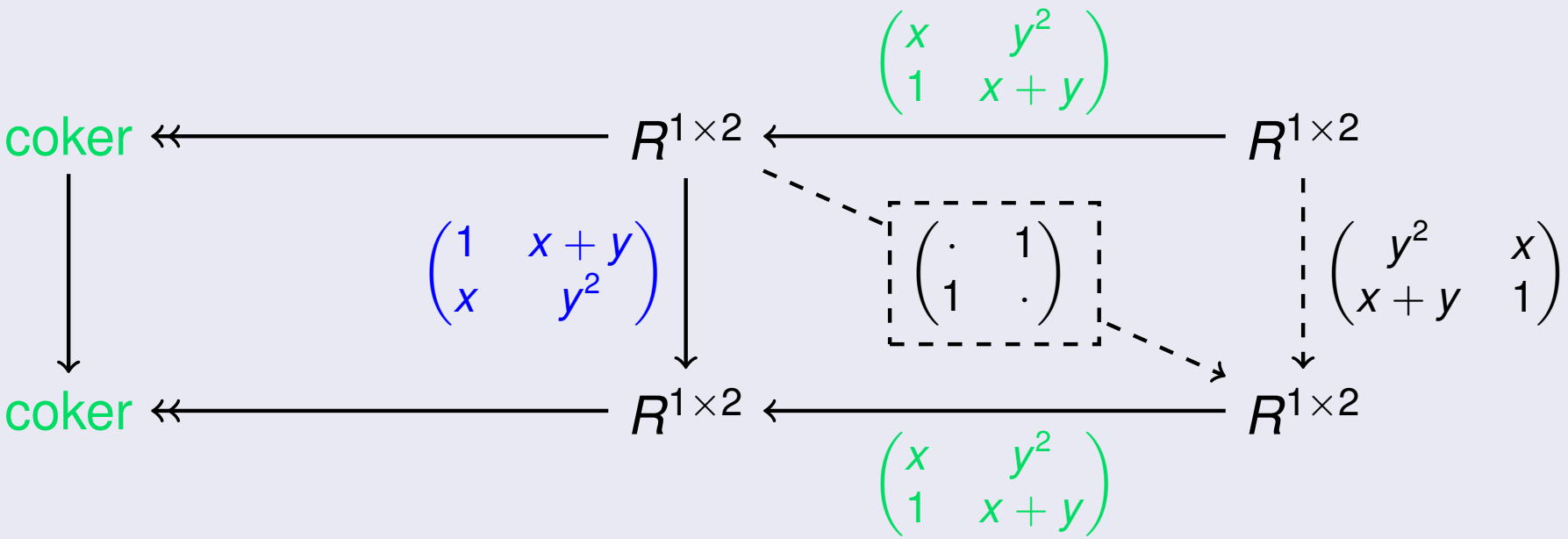

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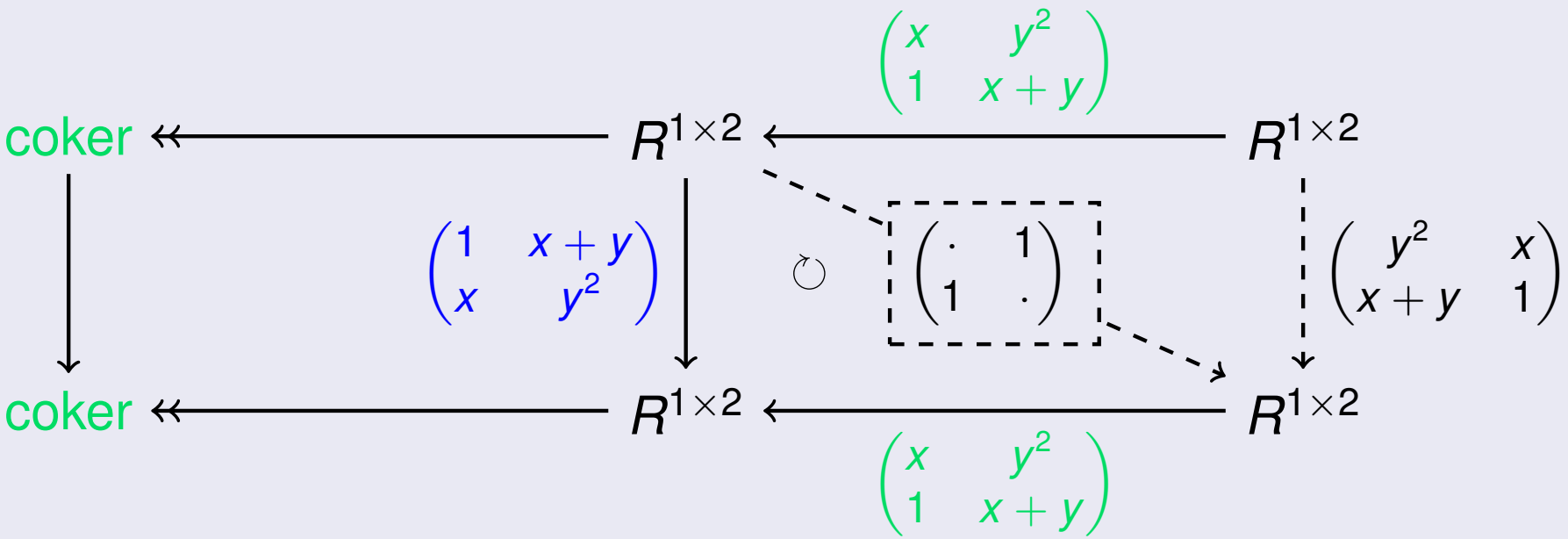

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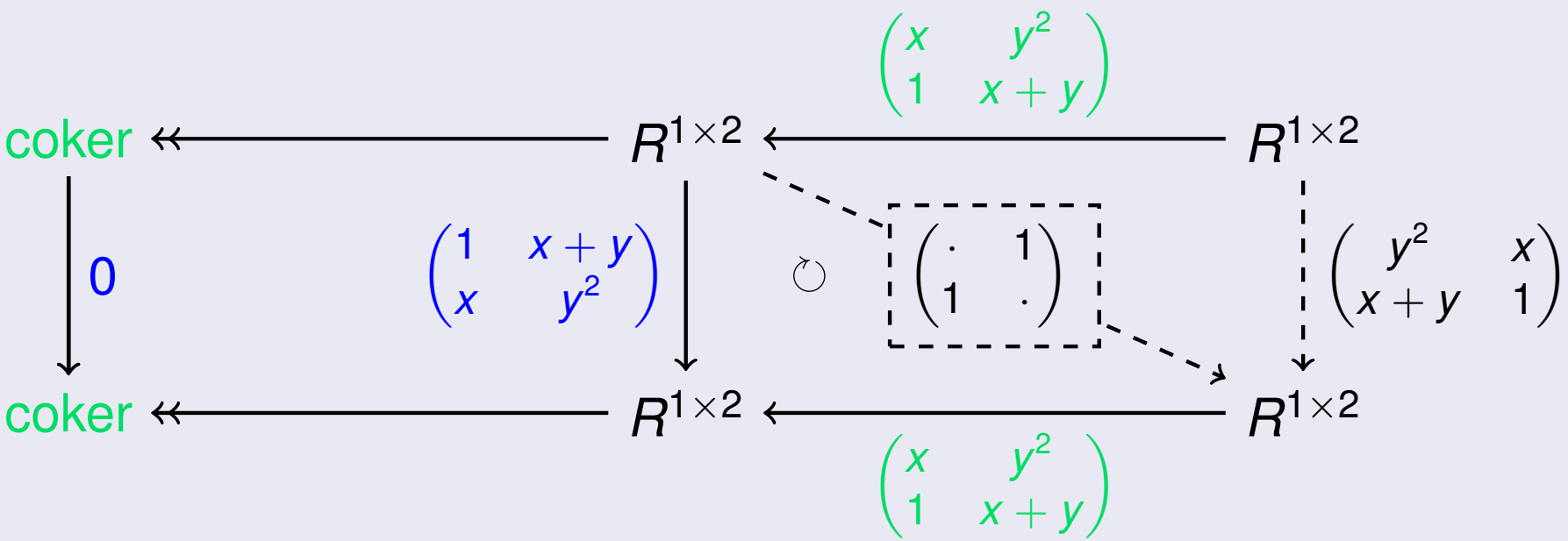

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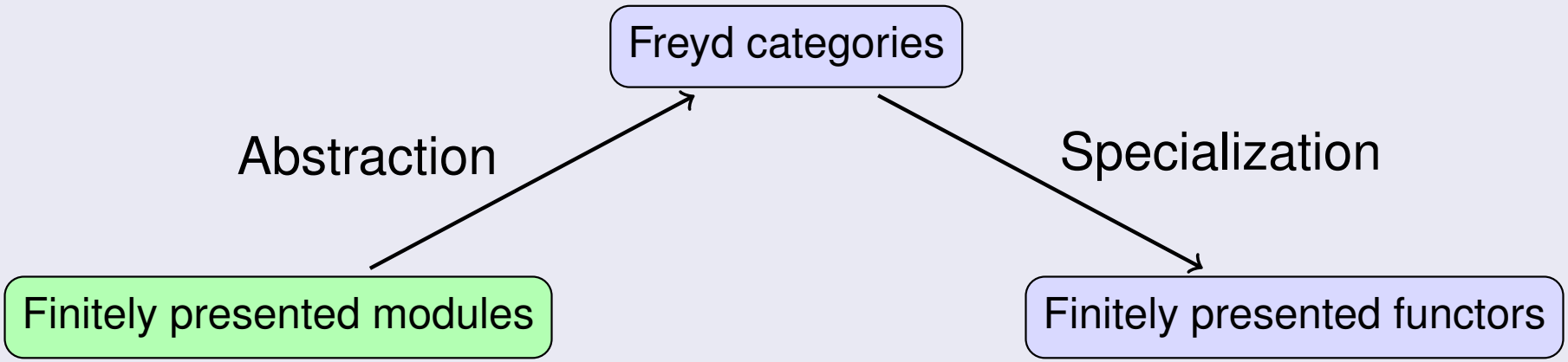

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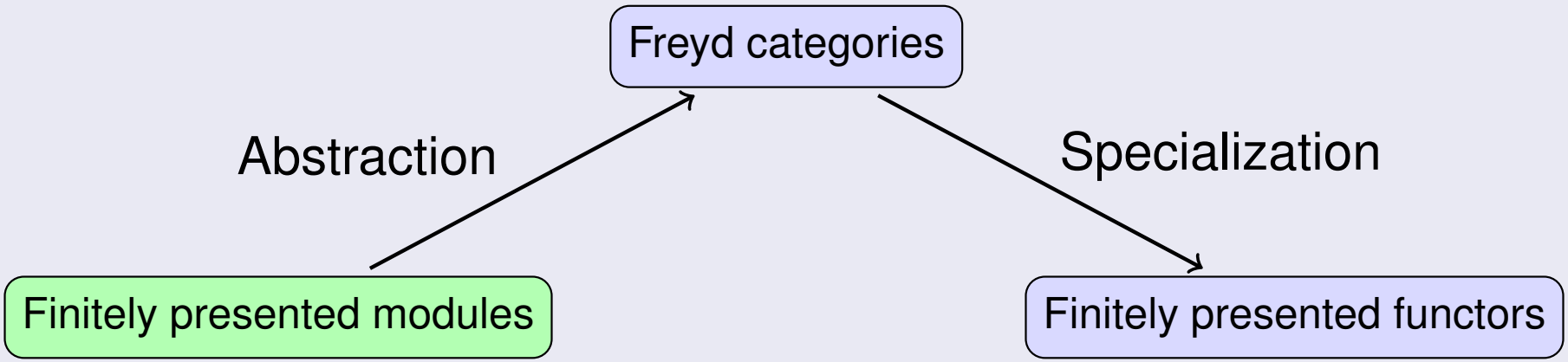

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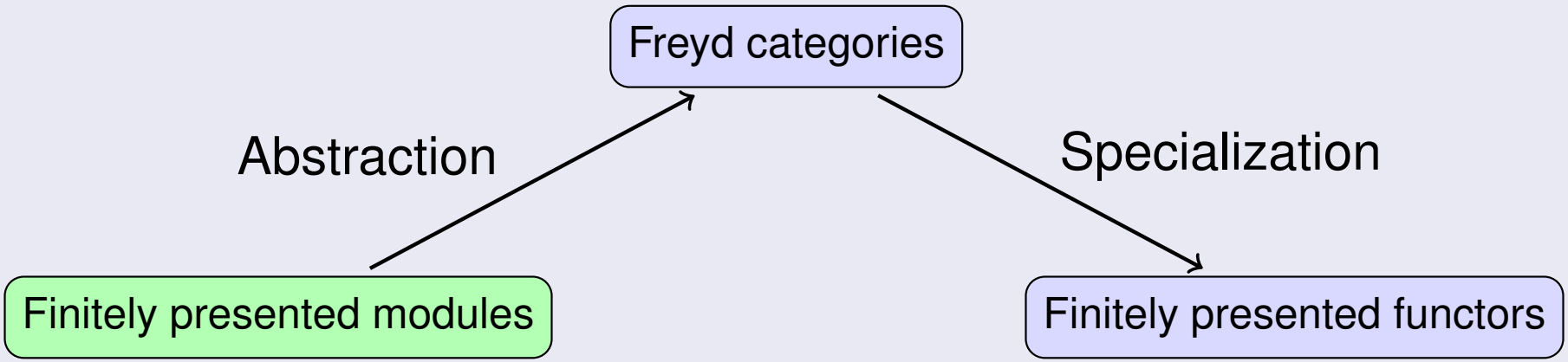

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Finitely presented modules: Algorithms

Algorithms in R-fpmod

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Algorithms for the defining axioms of an abelian category.

Finitely presented modules: Algorithms

Theorem (Barakat, Lange-Hegermann)

R-fpmod is computable as an abelian category if we have algorithms for the following two problems:

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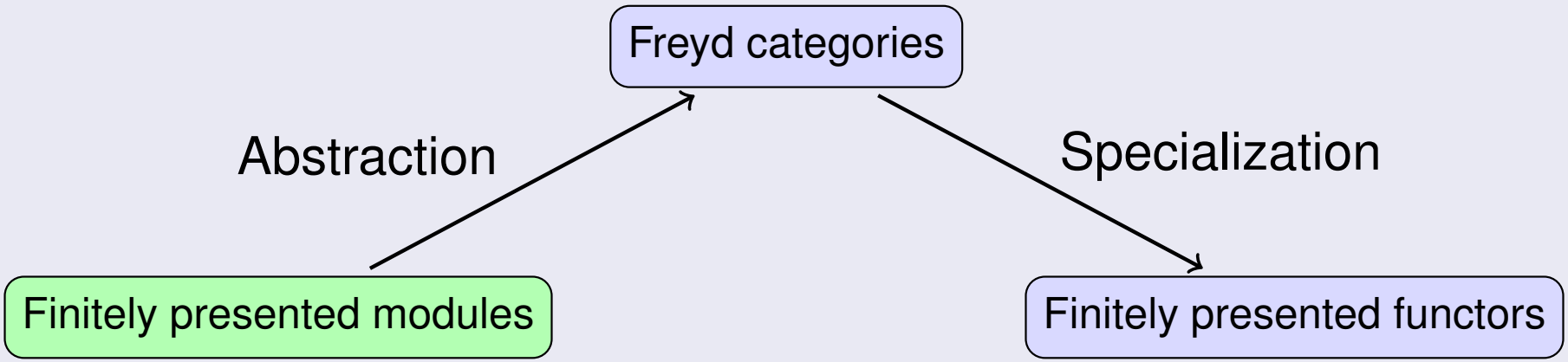

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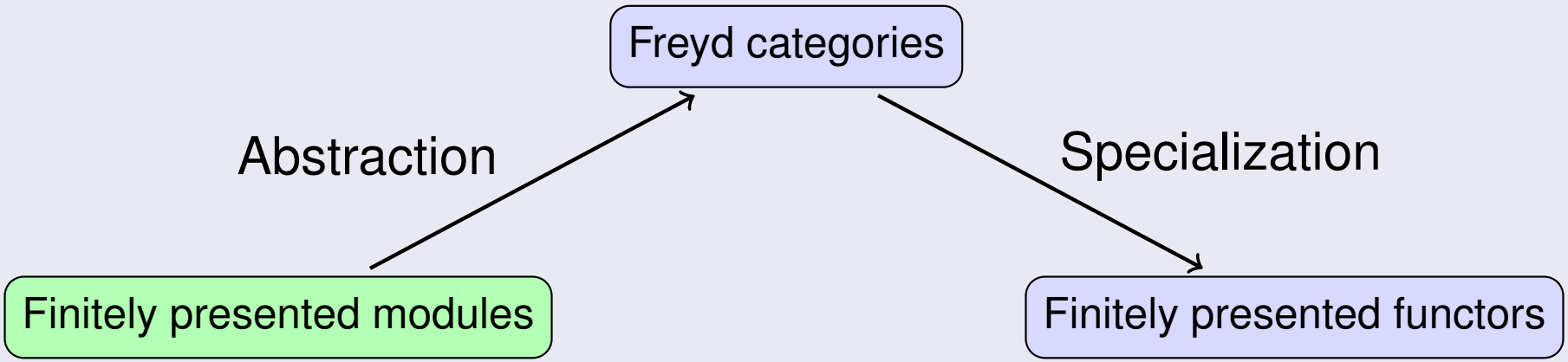

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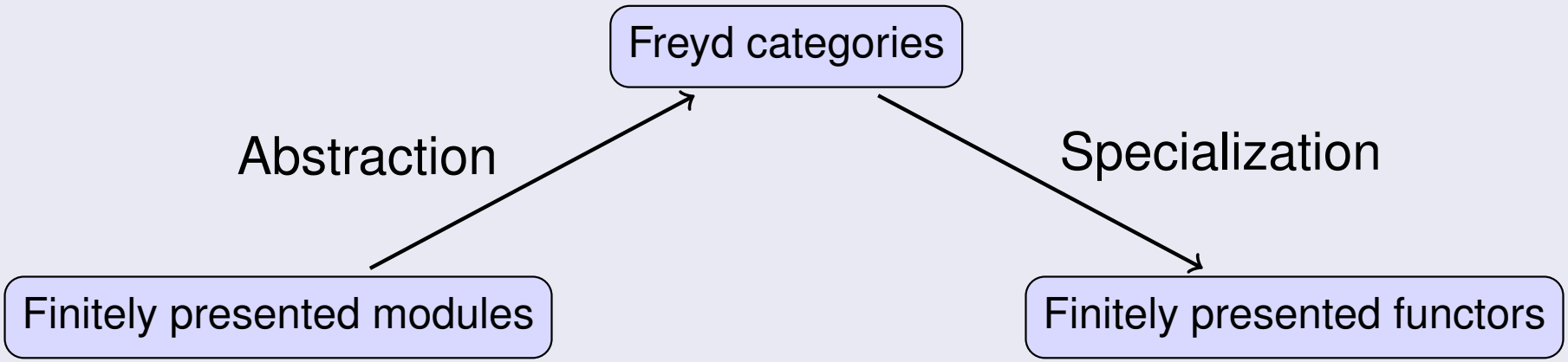

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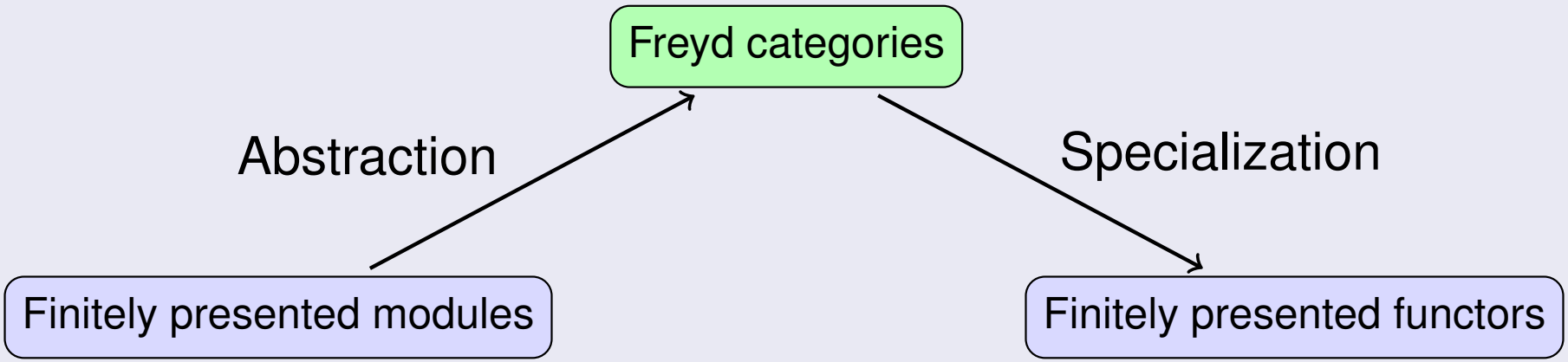

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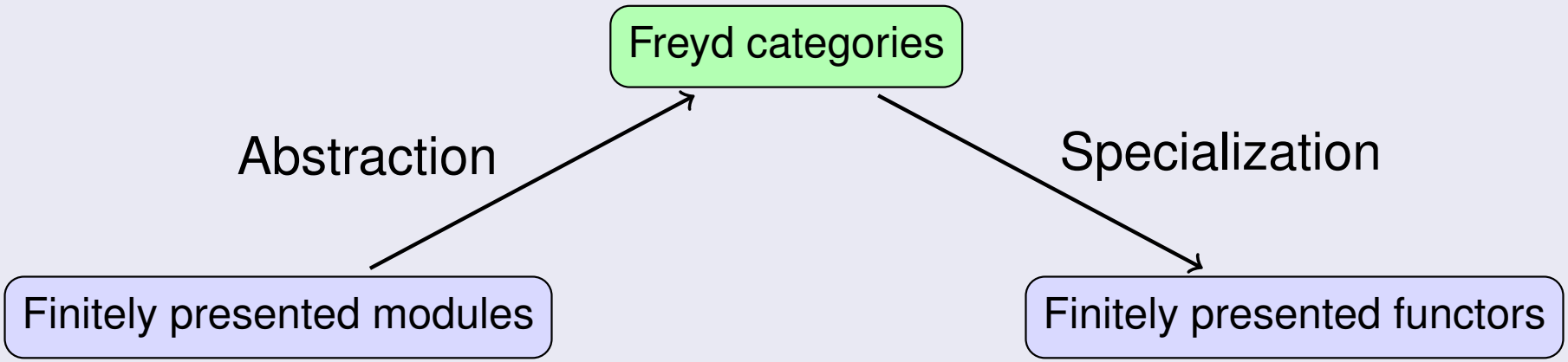

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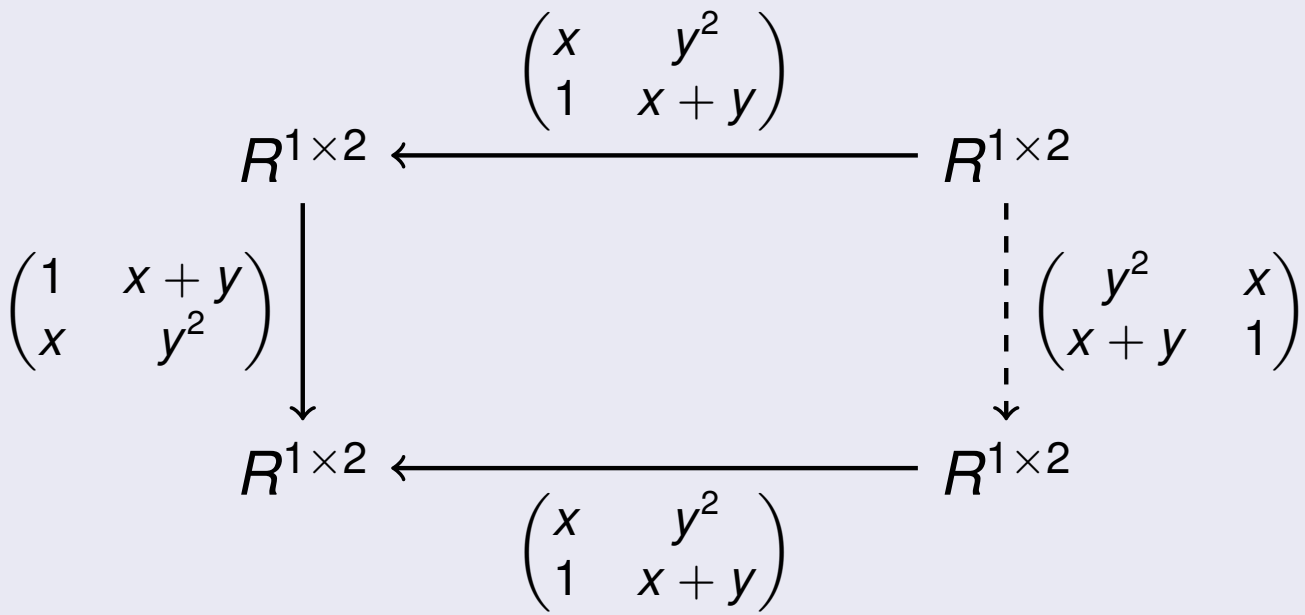
Freyd categories: Data structures

$R = \mathbb{Q}[x, y]$.

Category of finitely presented modules: R-fpmod



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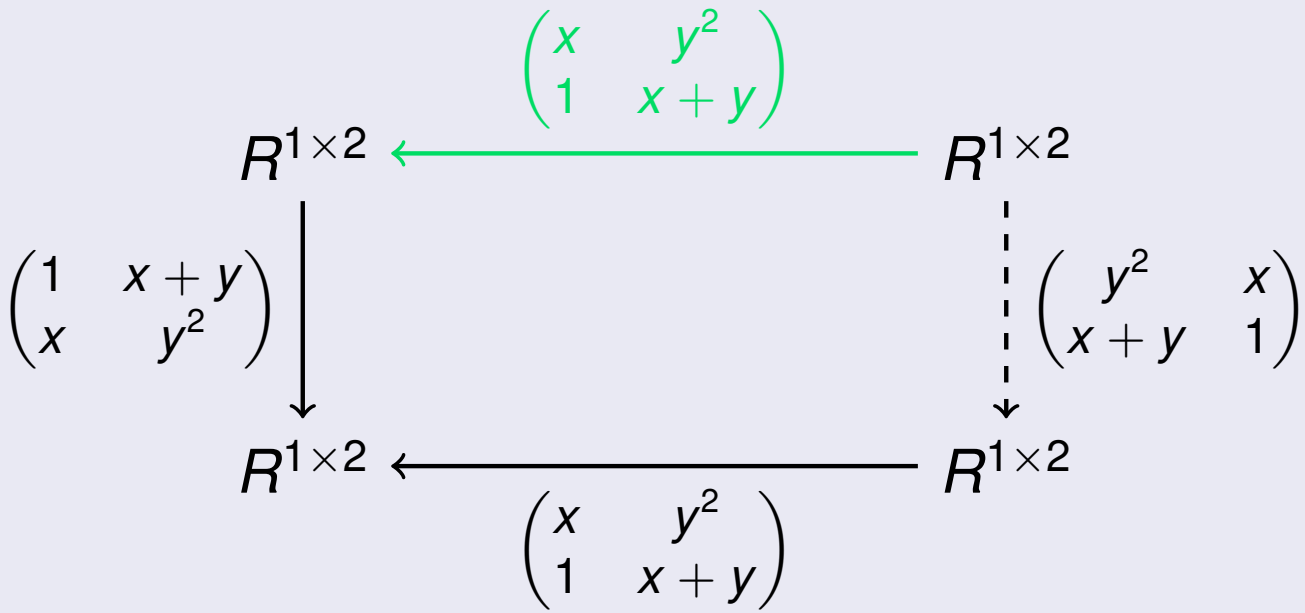

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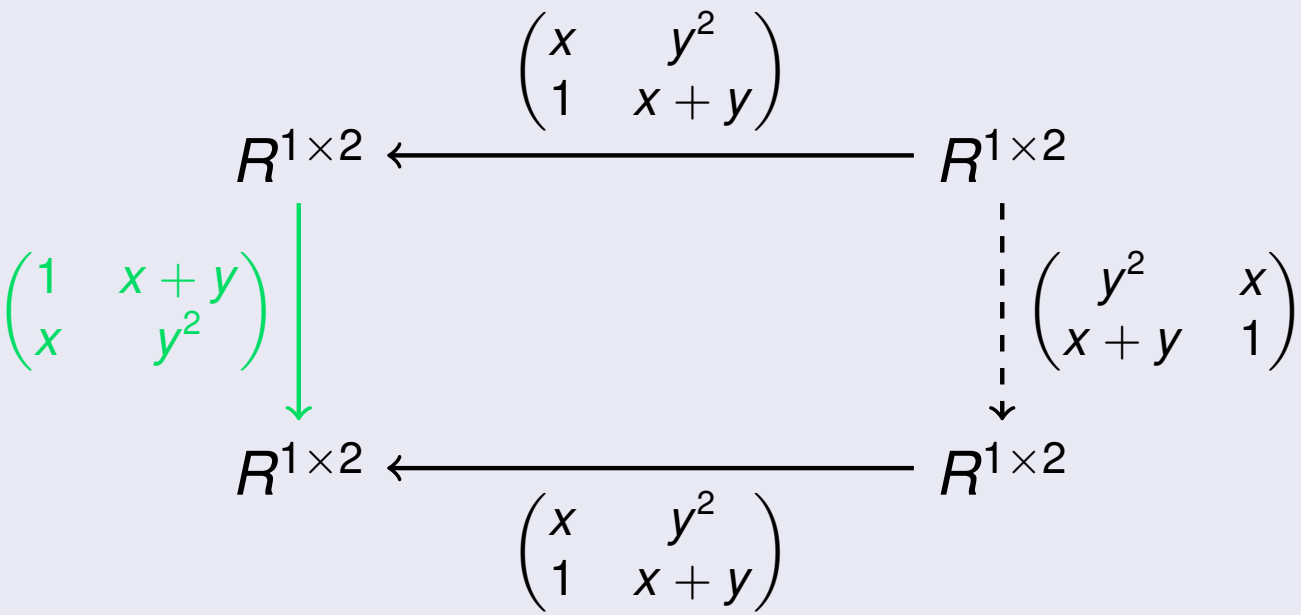

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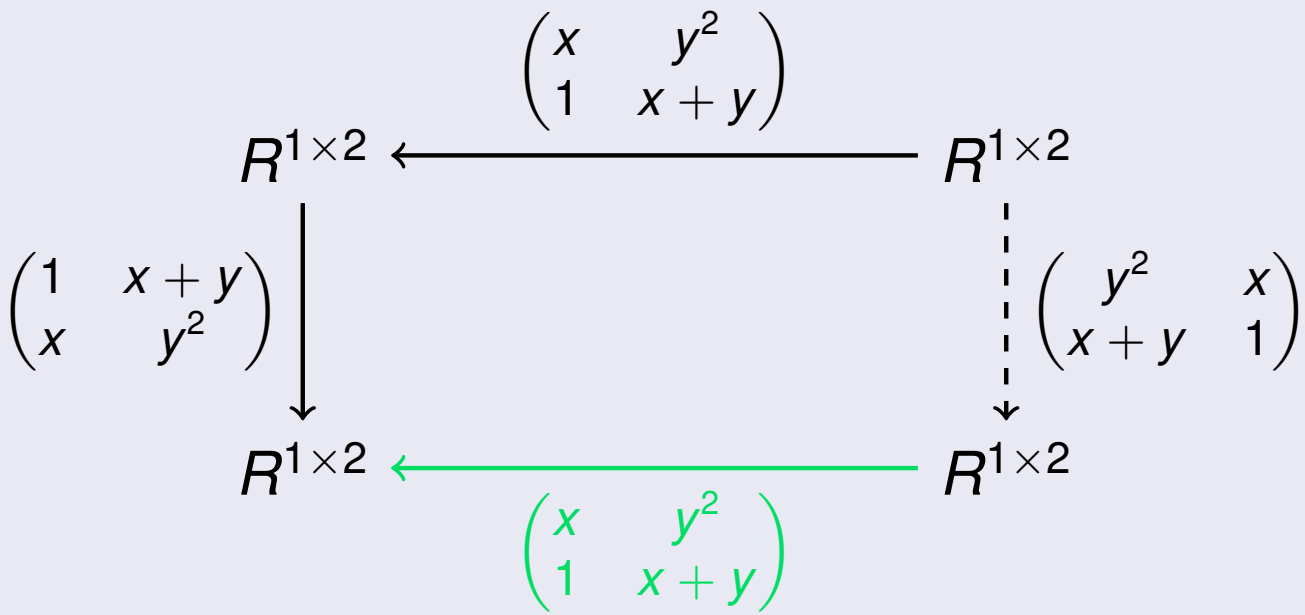

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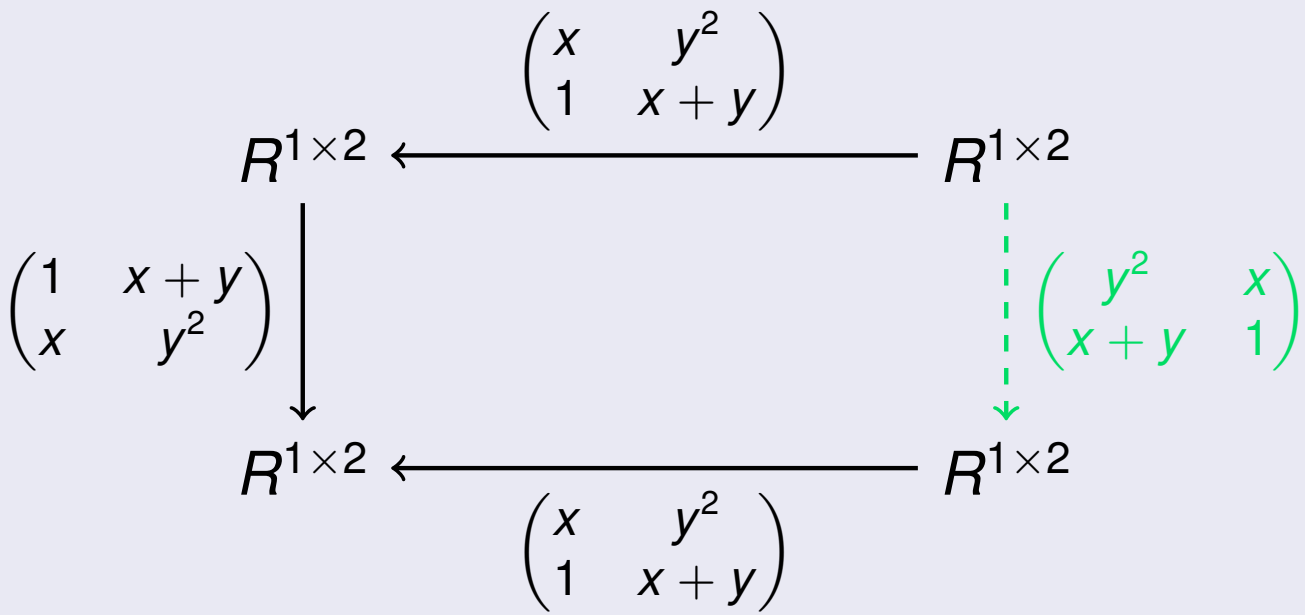

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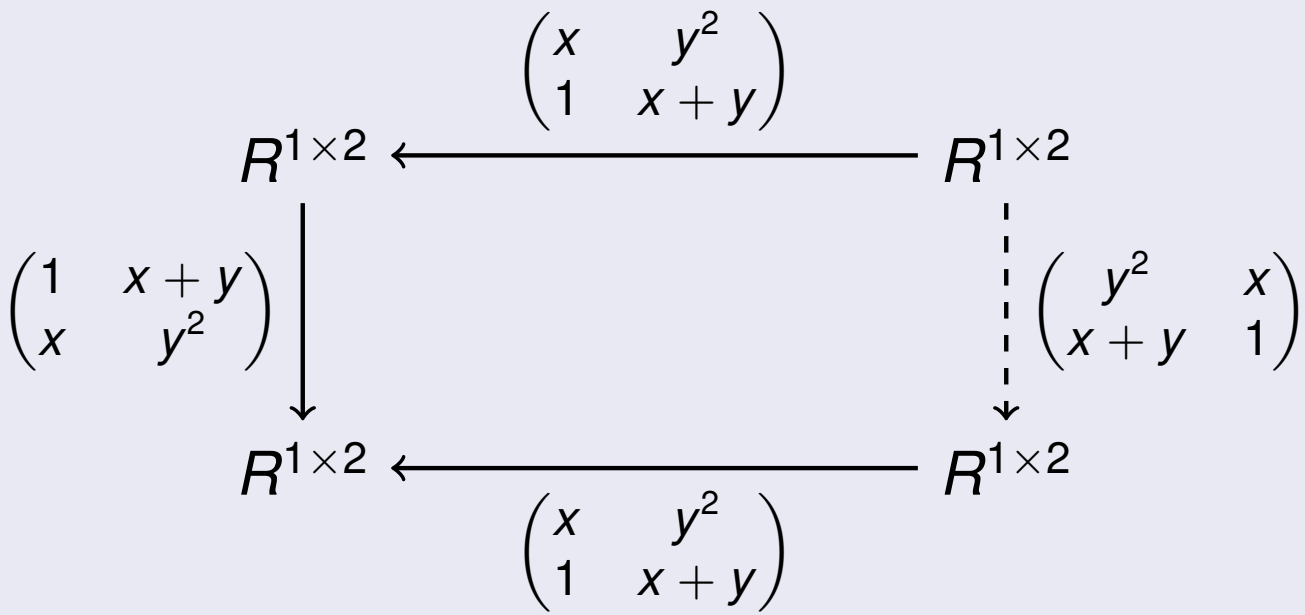

Observation: Data structures are built from objects and morphisms in

$$\mathrm{Rows}_R := \{R^{1\times n} \xleftarrow{A} R^{1\times m} \mid m, n \in \mathbb{N}_0, A \in R^{m\times n}\}.$$

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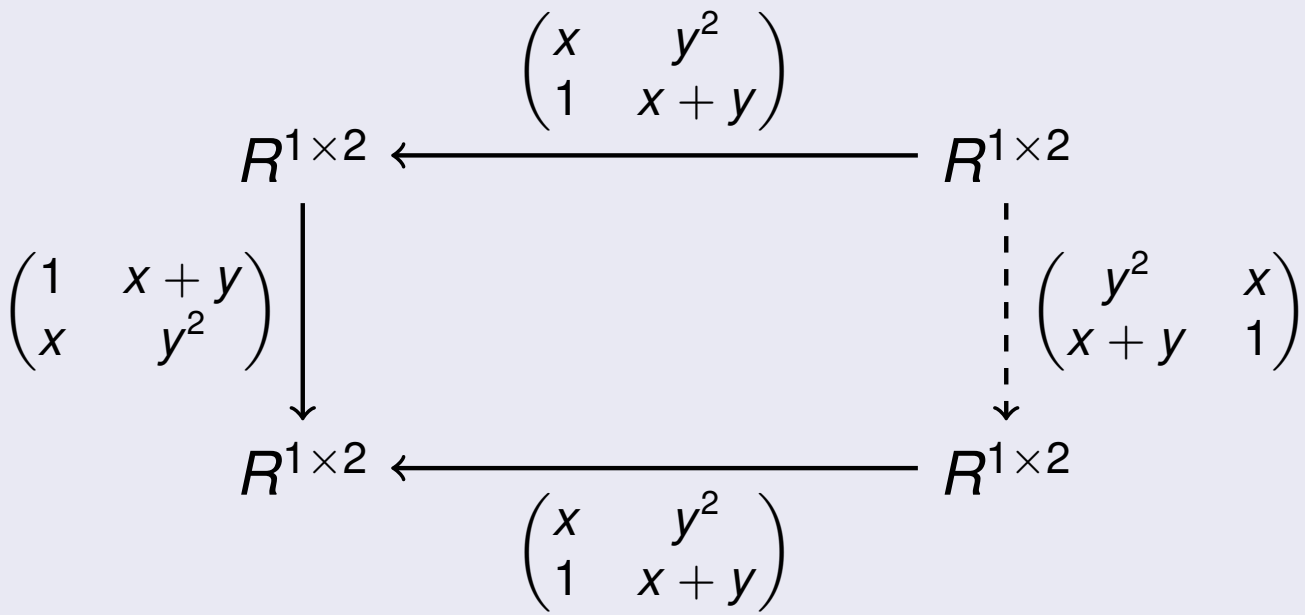

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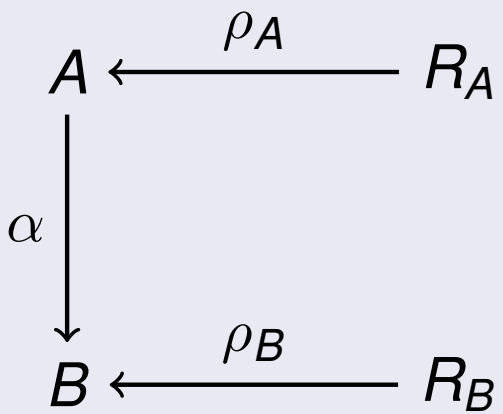

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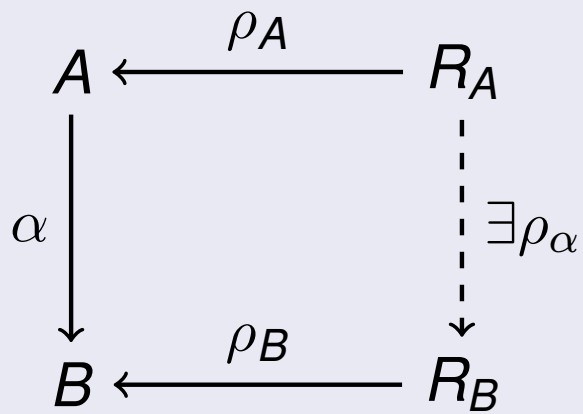

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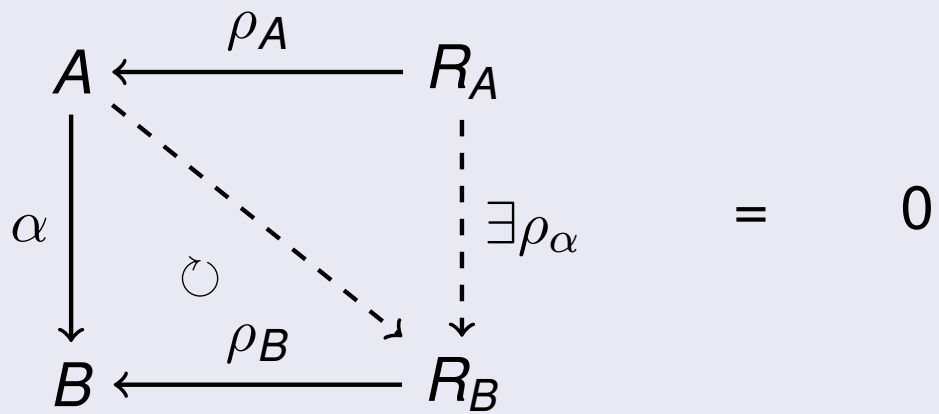

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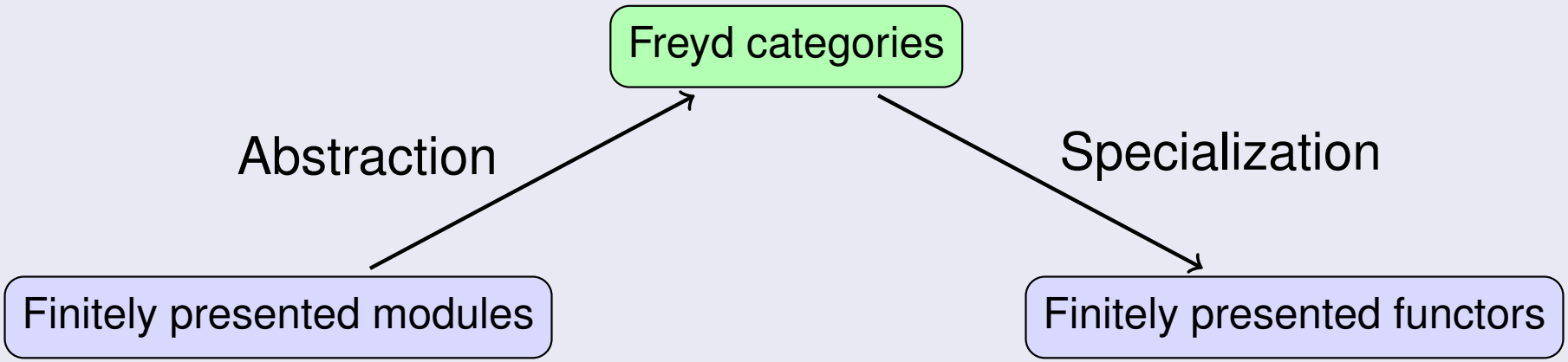

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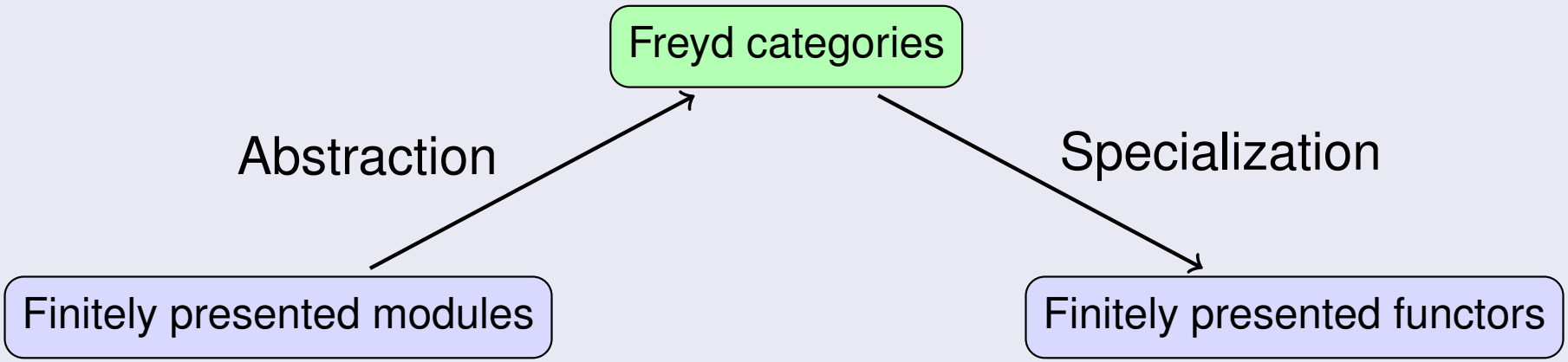

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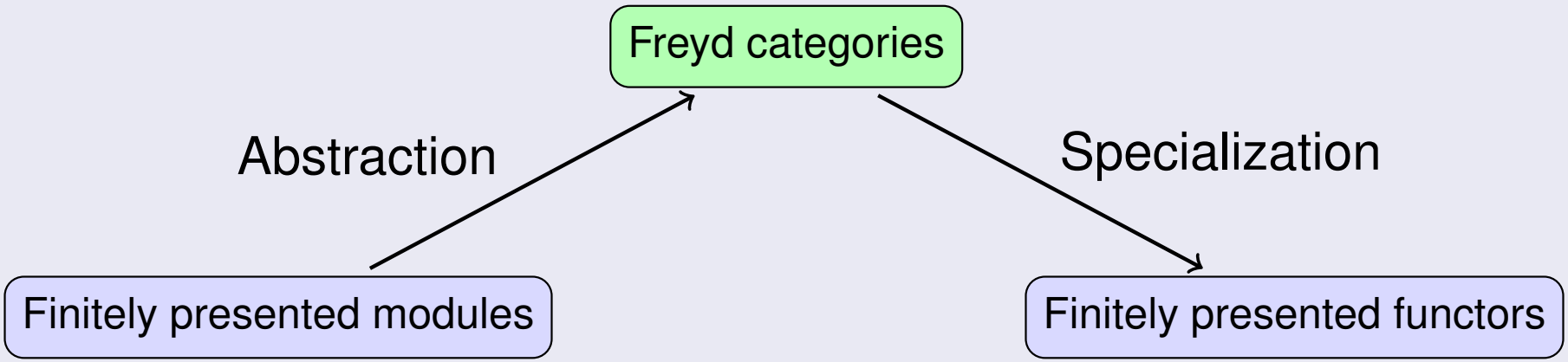

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R-fpmod is computable as an abelian category if we have algorithms for the following two problems:

- Given $A \in R^{m \times n}$. Find **syzygies**: $L \cdot A = 0$.
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What are syzygies in Rows_R?

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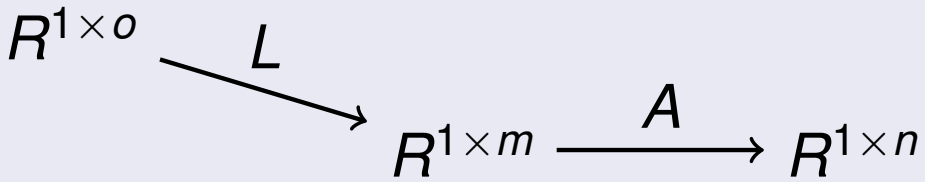

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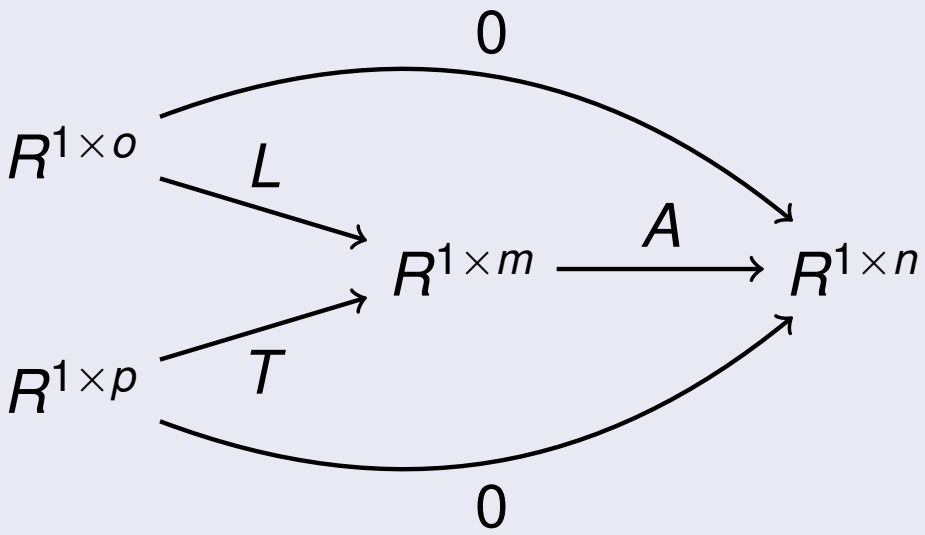

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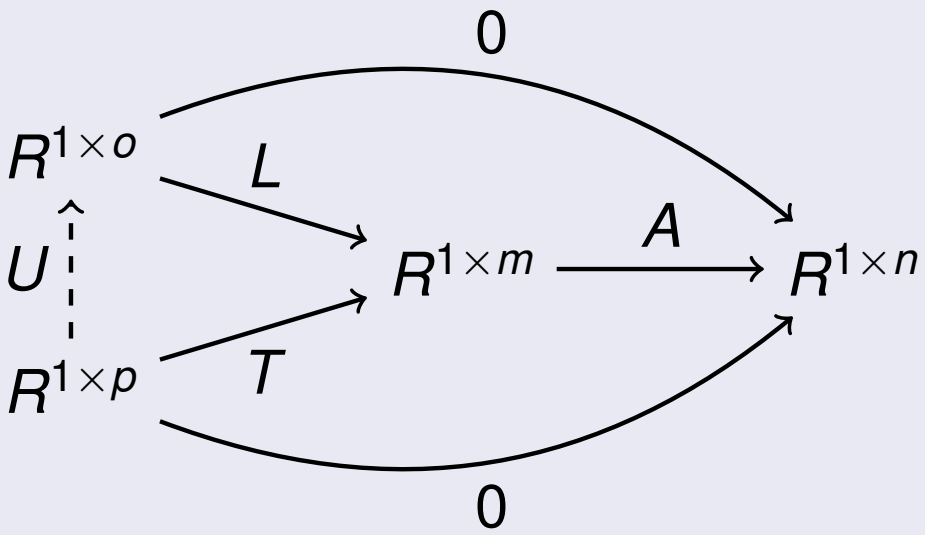

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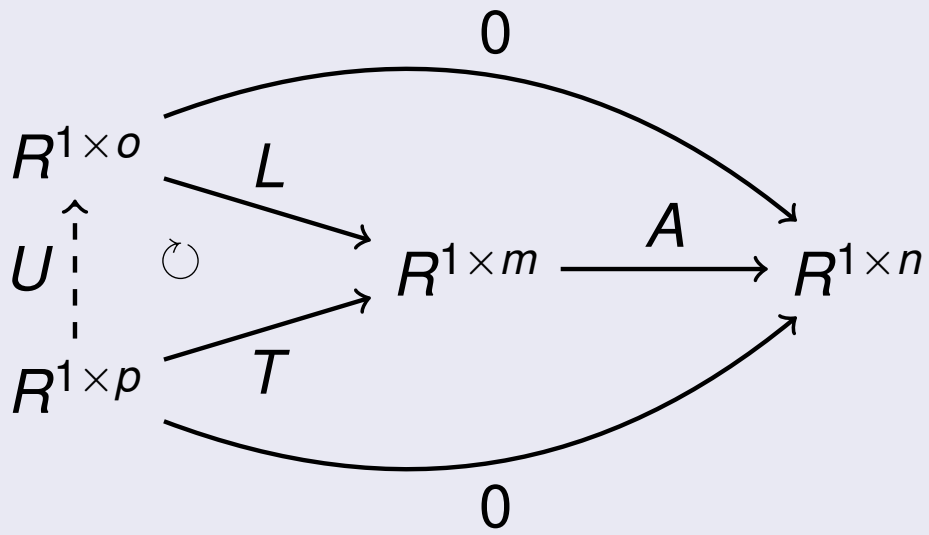

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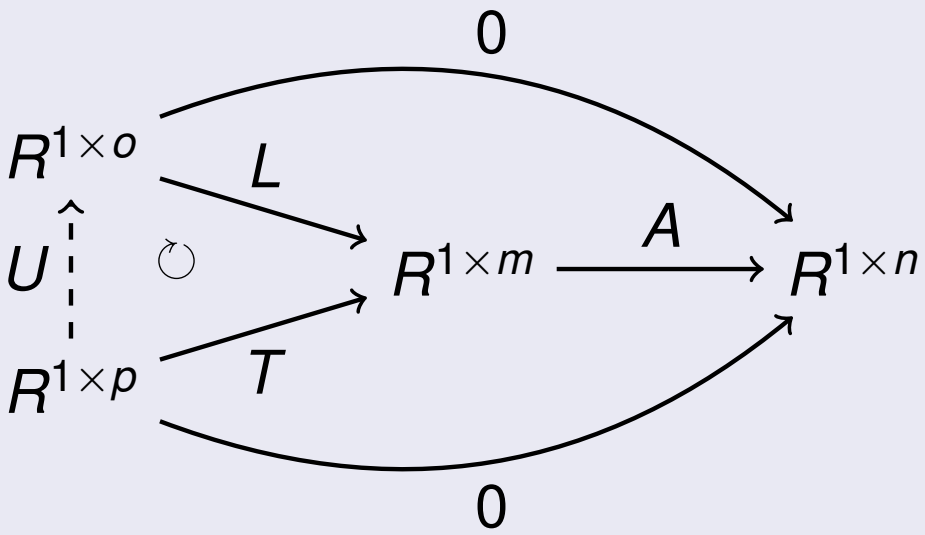

Syzygies = Weak kernels

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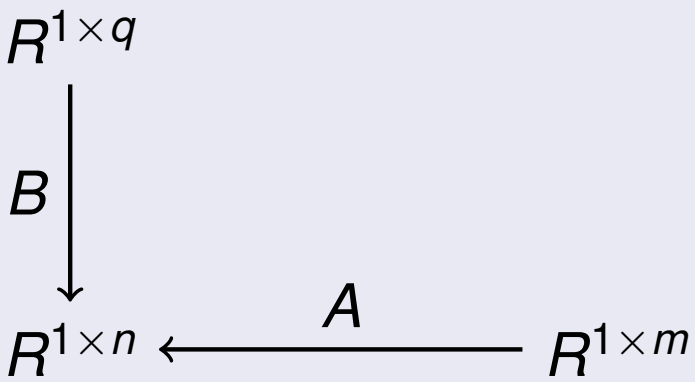

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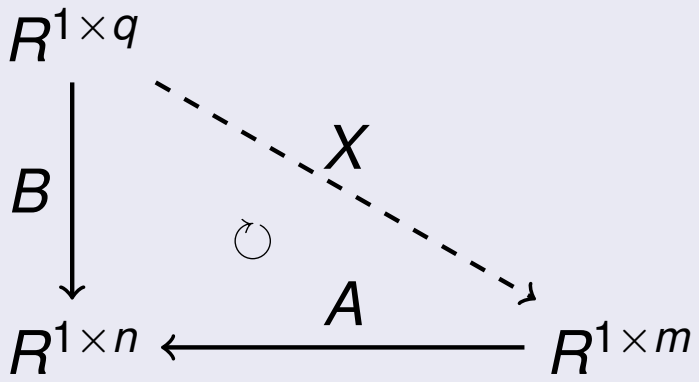

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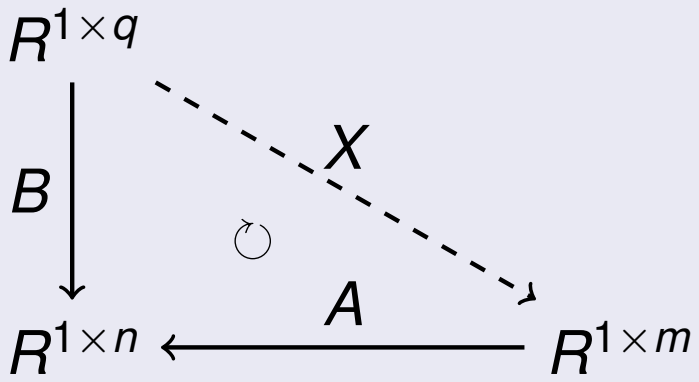

Solution = Lift of B along A.

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This theorem generalizes to Freyd categories.

Freyd categories: Algorithms

Theorem

Let **P** be a computable additive $(+, \oplus)$ category. The Freyd category $\mathcal{A}(\mathbf{P})$ is computable as an abelian category if and only if we have algorithms for the following two problems:

- Given $\alpha \in \mathrm{Mor}(\mathbf{P})$. Find **weak kernel** of α.
- Given $M \xrightarrow{\alpha} N$ and $Q \xrightarrow{\beta} N$ in $\mathrm{Mor}(\mathbf{P})$. Decide existence and in the affirmative case find **lift** of β along α.

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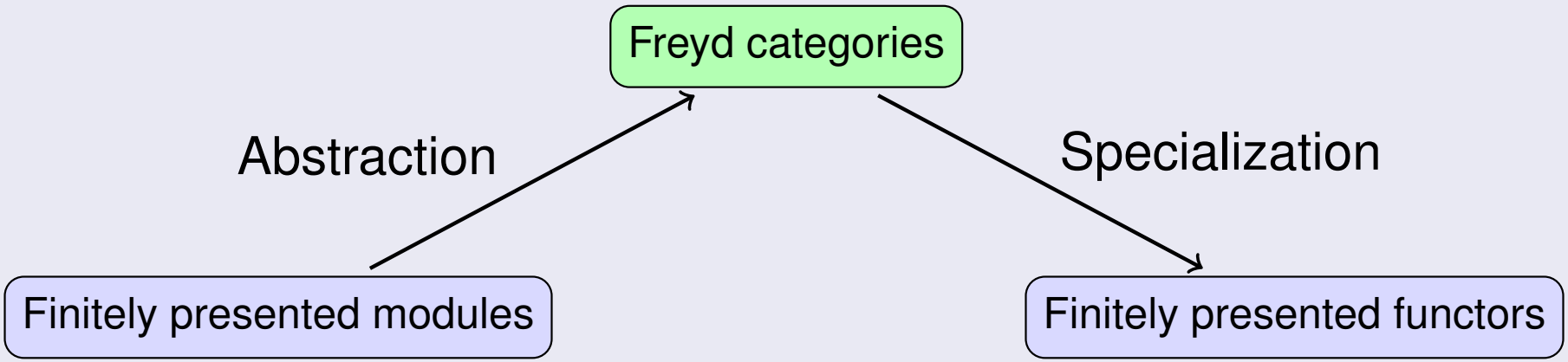
This is a constructive version of a theorem by Peter Freyd.

Route



1. Data structures ✓
2. Algorithms

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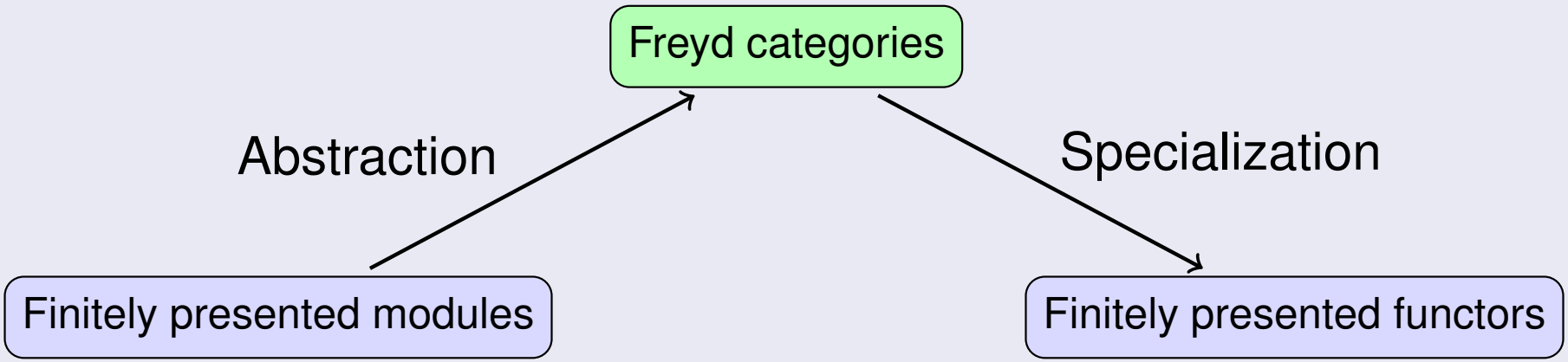

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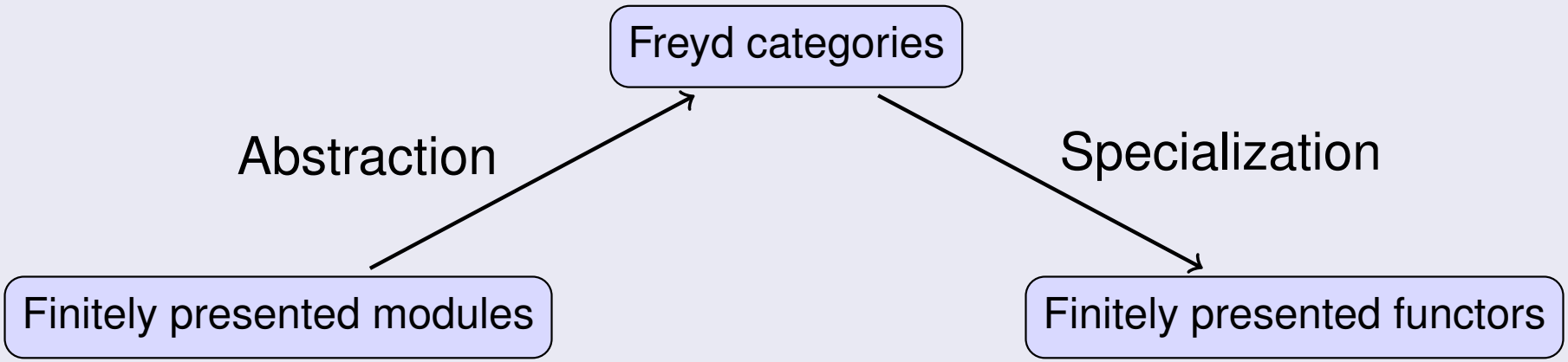

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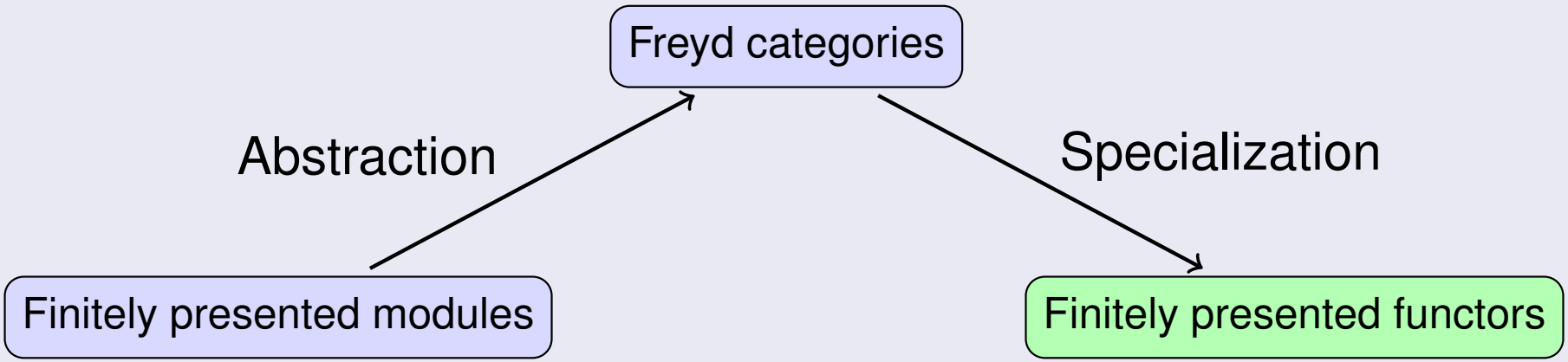

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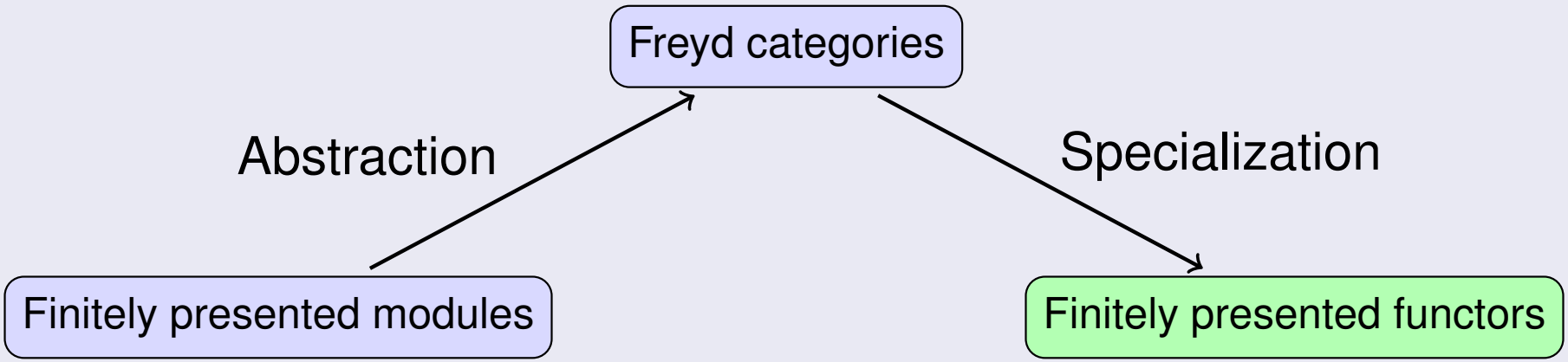

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Finitely presented functors: Data structures

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A **co**variant functor $F : R\text{-fpmod} \to \mathbf{Ab}$ is called **finitely presented** if there exist $M, N \in R\text{-fpmod}$ and an exact sequence of functors

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Thus, $\mathbf{P} \simeq R\text{-fpmod}^{\mathrm{op}}$.

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Remark

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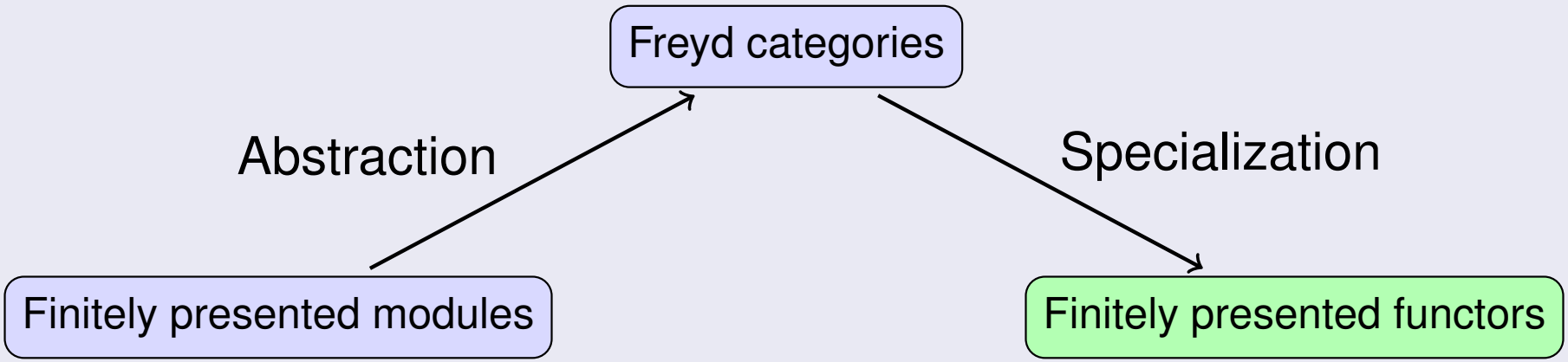

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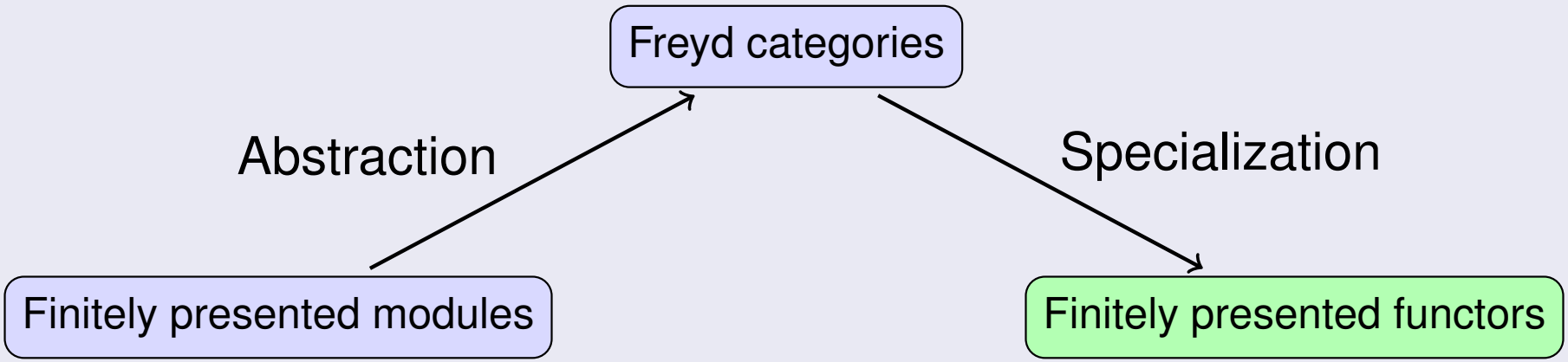

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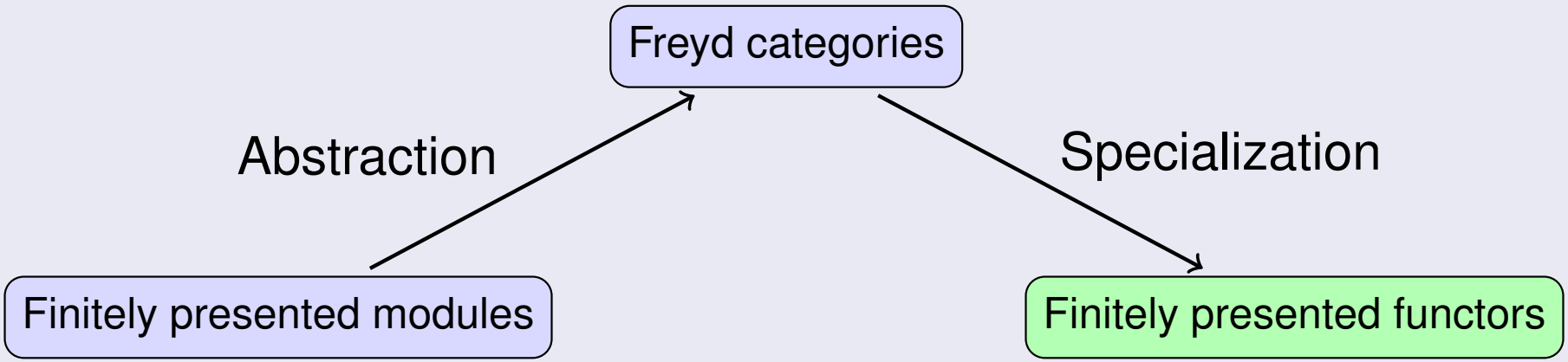

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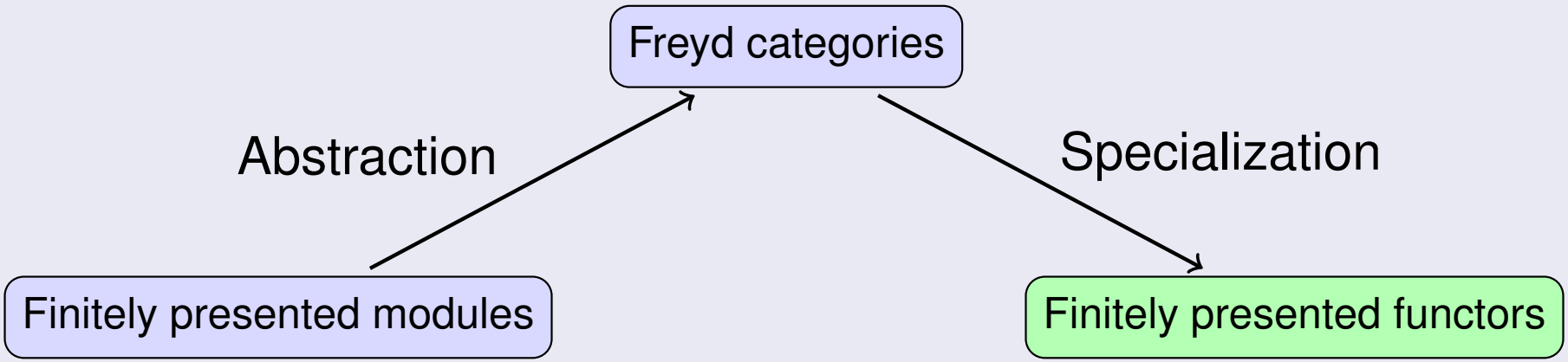

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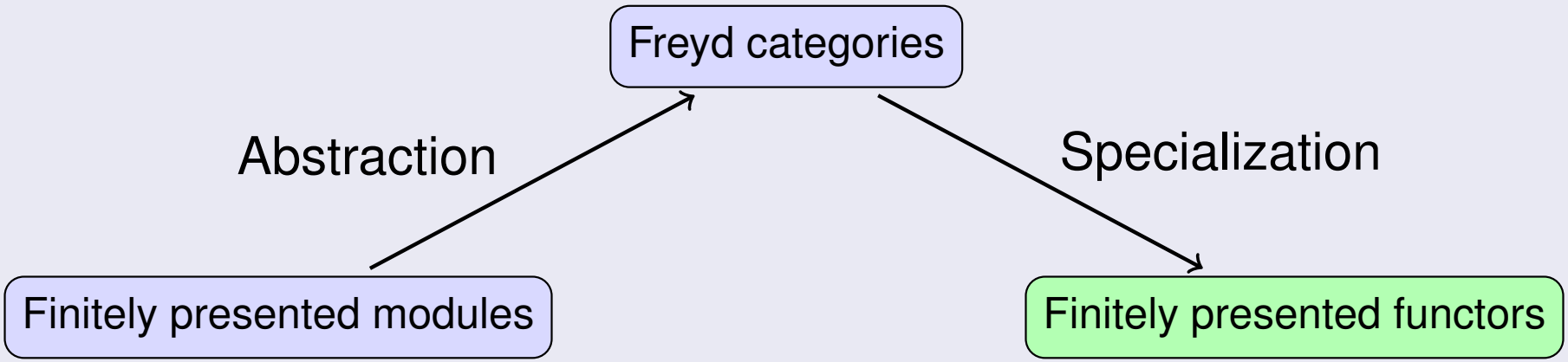

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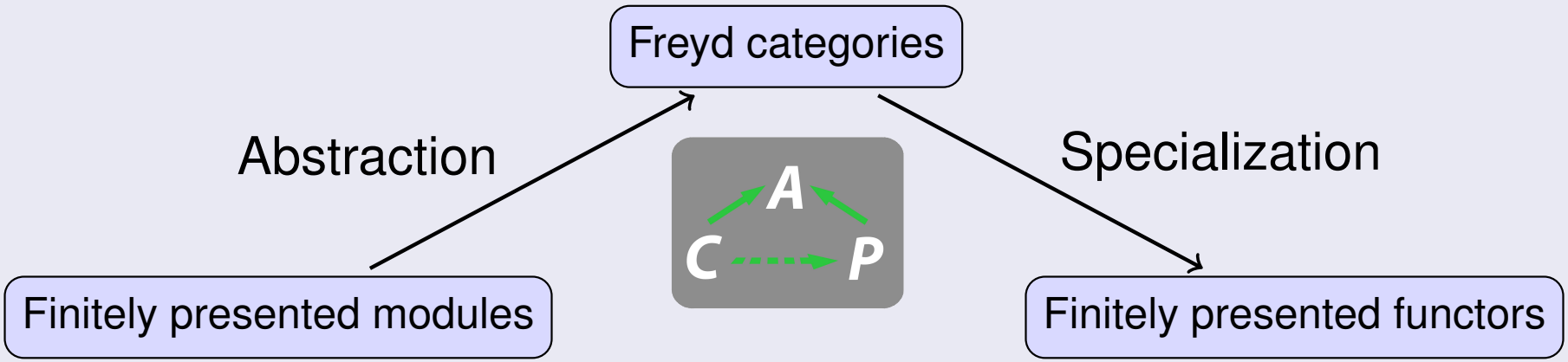

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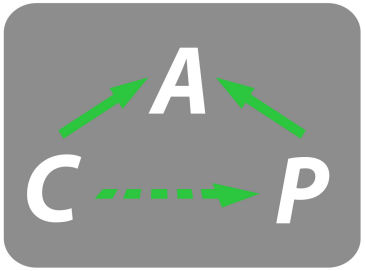
The CAP-project



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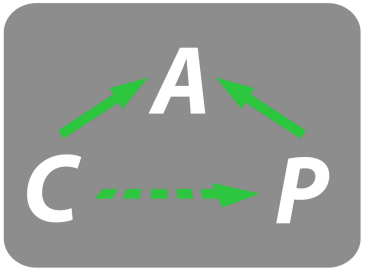
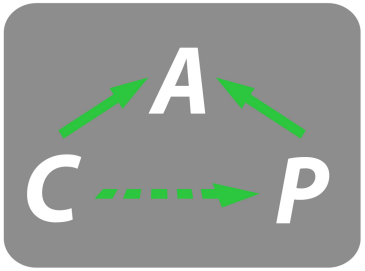


CAP: categories, algorithms, programming

category constructor	application

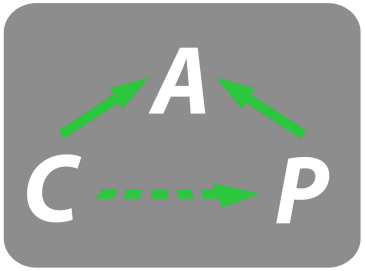
The CAP-project



CAP: categories, algorithms, programming

category constructor	application
$(\mathbf{A}, \mathbf{C}) \mapsto \mathbf{A}/\mathbf{C}$	

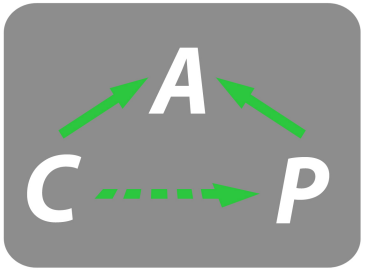
The CAP-project



CAP: categories, algorithms, programming

category constructor	application
$(\mathbf{A}, \mathbf{C}) \mapsto \mathbf{A}/\mathbf{C}$	coherent sheaves on toric varieties

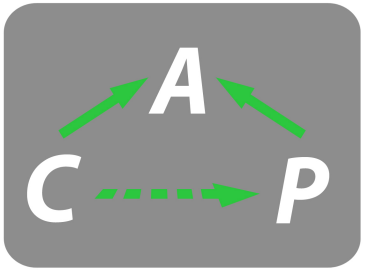
The CAP-project



CAP: categories, algorithms, programming

category constructor	application
$(\mathbf{A}, \mathbf{C}) \mapsto \mathbf{A}/\mathbf{C}$	coherent sheaves on toric varieties
$\mathbf{A} \mapsto \mathrm{Ch}^\bullet(\mathbf{A})$	

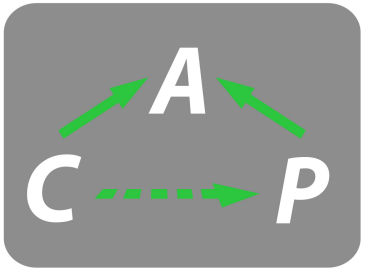
The CAP-project



CAP: categories, algorithms, programming

category constructor	application
$(\mathbf{A}, \mathbf{C}) \mapsto \mathbf{A}/\mathbf{C}$	coherent sheaves on toric varieties
$\mathbf{A} \mapsto \mathrm{Ch}^\bullet(\mathbf{A})$	towards dg categories

The CAP-project



CAP: categories, algorithms, programming

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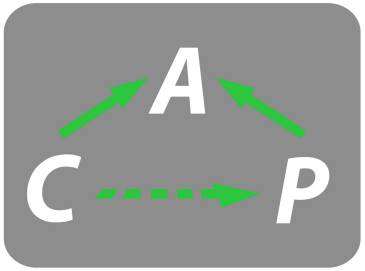
$\mathbf{A} \mapsto \mathrm{Ch}^\bullet(\mathbf{A})$	towards dg categories
$\mathbf{A} \mapsto \mathbf{G}(\mathbf{A})$	

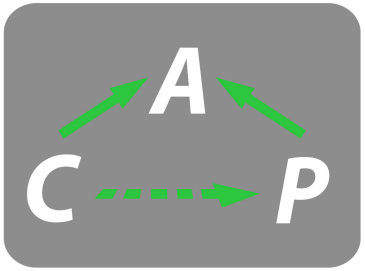
The CAP-project



CAP: categories, algorithms, programming

category constructor	application
$(\mathbf{A}, \mathbf{C}) \mapsto \mathbf{A}/\mathbf{C}$	coherent sheaves on toric varieties
$\mathbf{A} \mapsto \mathrm{Ch}^\bullet(\mathbf{A})$	towards dg categories
$\mathbf{A} \mapsto \mathbf{G}(\mathbf{A})$	diagram chasing

The CAP-project



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$(\mathbf{A}, \mathbf{B}) \mapsto \mathrm{Hom}(\mathbf{A}, \mathbf{B})$	

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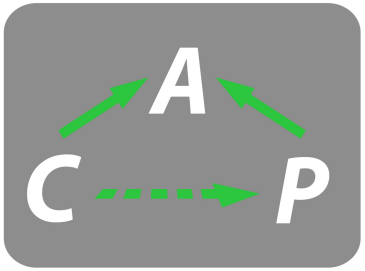

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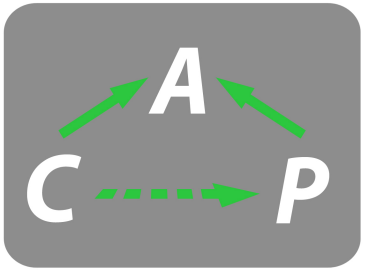
$\mathbf{A} \mapsto \mathrm{Ch}^\bullet(\mathbf{A})$	towards dg categories
$\mathbf{A} \mapsto \mathbf{G}(\mathbf{A})$	diagram chasing
$(\mathbf{A}, \mathbf{B}) \mapsto \mathrm{Hom}(\mathbf{A}, \mathbf{B})$	representation theory of f.d. algebras

The CAP-project



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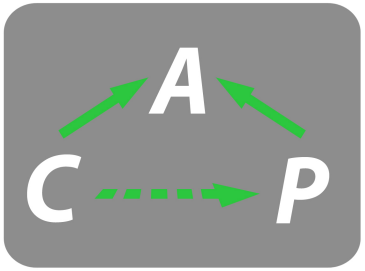
$(\mathbf{A}, \mathbf{B}) \mapsto \mathrm{Hom}(\mathbf{A}, \mathbf{B})$	representation theory of f.d. algebras
$\mathbf{P} \mapsto \mathcal{A}(\mathbf{P})$	

The CAP-project



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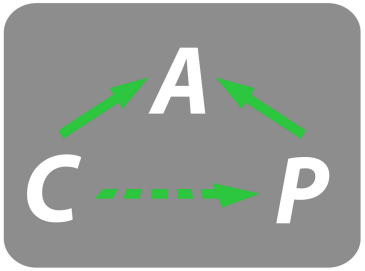
$\mathbf{A} \mapsto \mathbf{G}(\mathbf{A})$	diagram chasing
$(\mathbf{A}, \mathbf{B}) \mapsto \mathrm{Hom}(\mathbf{A}, \mathbf{B})$	representation theory of f.d. algebras
$\mathbf{P} \mapsto \mathcal{A}(\mathbf{P})$	f.p. functors

The CAP-project



CAP: categories, algorithms, programming

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$\mathbf{P} \mapsto \mathcal{A}(\mathbf{P})$	f.p. functors

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